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BENDING OF A RIB, SUPPORTING THE EDGE OF A PLANE THIN BODY, BY A CONCENTRATED FORCE

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We propose simple asymptotic formulas for calculating improper integrals in the zone of application of a local load, where the highest-power stressed state develops. It was possible to construct these formulas by expanding quadratures into a power series (multiplied by a logarithmic function) that converges rapidly at small values of the argument.

Introduction. The places of application of local loads to thin-walled structural elements are usually reinforced by cover plates and stiffening ribs. Therefore, investigation of the influence of reinforcement on the stress-strain state of both the supporting element and the structure (in the present case a plane body) is of practical interest. It is assumed that an infinitely long rib that has a finite bending stiffness, reinforces the edge of a half-plane and is loaded by a concentrated force located at the coordinate origin (see Fig. 1). The two bodies interact only in the direction perpendicular to the edge of the plane body, as a result of which only contact forces that lie in the plane of the body and are transverse to the edge appear. Tangential components of possible interaction are neglected. This formulation of the problem is typical of a number of publications noted in [1]. Mathematically, this formulation is reduced to the Fourier integrals that appear in the problem of cylindrical bending of a plate on an elastic half-space [2-4]. In order to simplify the solutions and impart to them the corresponding differential properties, the convergence of these integrals was accelerated [4], but we failed to eliminate them completely.

1. Analysis of Improper Integrals by the Watson Method. Leaving aside the procedure for constructing the solution, which can be found, for example, in [3], we will write out integrals for calculating the contact reactions $q(y)$, angle of rotation of the rib, bending moment $M(y)$, and transverse force $Q(y)$ that arise in the rib:

$$q(y) = P\pi^{-1} a^3 \int_0^\infty \Delta(\beta) \cos \beta y \, d\beta, \quad w'(y) = P(\pi E_1 I_1)^{-1} \int_0^\infty \Delta(\beta) \sin \beta y \, d\beta,$$
$$M(y) = P\pi^{-1} \int_0^\infty \beta \Delta(\beta) \cos \beta y \, d\beta, \quad Q(y) = -P\pi^{-1} \int_0^\infty \beta^2 \Delta(\beta) \sin \beta y \, d\beta, \tag{1.1}$$

where $\Delta(\beta) = (\beta^3 + a^3)^{-1}$; $a = (Eh/2E_1 I_1)^{1/3}$.

Let us consider in more detail the calculation of the contact force $q(y)$. We apply a method proposed by G. Watson [5] for calculating integrals of cylindrical functions and use the Mellin–Berns integral representation for the trigonometric function [6]

$$\cos \beta y = \frac{1}{4i} \int_{c-i\infty}^{c+i\infty} \frac{(\beta y)^z \, dz}{\Gamma(1+z) \sin \frac{\pi z}{2}}, \tag{1.2}$$

where $\Gamma(z)$ is the gamma-function; $i = \sqrt{-1}$; $-1/2 < c < 0$; $\mathrm{Re}\, z \ge c$.

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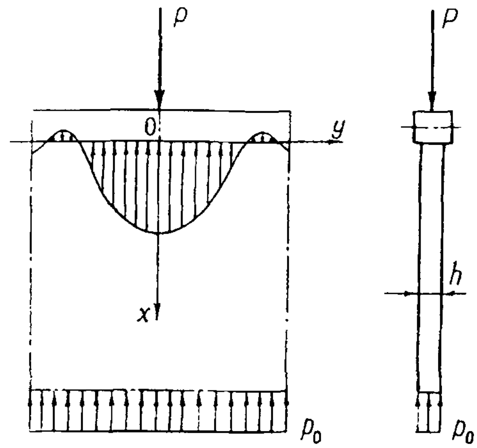

Fig. 1. Plane body loaded through a supporting rib fastened to it and having a finite bending stiffness.

Integral (1.2) can be calculated by means of the residue theorem. To have a closed contour, we will supplement the straight line Re $z = c$ with a circular arc of radius R, locating it to the right of this straight line and letting R go to infinity. Clockwise tracing of the contour is taken to be the positive direction. Within the region Re $z \geq c$ the function $\Gamma(1+z)$ differs from zero, and only the simple poles $z = 2m$ ($m = 0, 1, 2, 3, \ldots$) are singular points. The sum of the residues at them, multiplied by $2\pi i$, leads to a Maclaurin series for $\cos \beta y$. Consequently, integral (1.2) over the circular arc of radius R tends to zero when $R \to \infty$, and expression (1.2) represents a trigonometric function.

Substituting Eq. (1.2) into Eq. (1.1) and rearranging the order of integration, we obtain

$$\int_0^\infty \Delta(\beta) \cos \beta y \, d\beta = \frac{1}{4i} \int_{c-i\infty}^{c+i\infty} \frac{y^z \, dz}{\Gamma(1+z) \sin \frac{\pi z}{2}} \int_0^\infty \beta^z \Delta(\beta) \, d\beta . \tag{1.3}$$

The inner integral in Eq. (1.3) can be found with the aid of the tables of [7], where

$$\int_0^\infty \frac{x^{\mu-1}}{(p+qx^\nu)^{n+1}} = \frac{1}{\nu p^{n+1}} \left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu)\,\Gamma(1+n-\mu/\nu)}{\Gamma(1+n)} . \tag{1.4}$$

Taking into account that [7]

$$\Gamma(\mu/\nu)\,\Gamma(1-\mu/\nu) = \pi \sin^{-1}(\pi \mu/\nu) ,$$

from expressions (1.3) and (1.4) we have

$$\int_0^\infty \Delta(\beta) \cos \beta y \, d\beta = \frac{\pi}{12a^2 i} \int_{c-i\infty}^{c+i\infty} \frac{(ay)^z \, dz}{\Gamma(1+z) \sin \frac{\pi z}{2} \sin\left(\pi \frac{z+1}{3}\right)} . \tag{1.5}$$

Integral (1.5) is taken by means of the residue theorem using the same contour as in calculating (1.2). Poles of the first and second orders are the singular points of the integrand. Simple ones are located on the real axis at $z = 6m$, $z = 6m + 4$, and $z = 6m + 5$, and multiple ones are located at $z = 6m + 2$ ($m = 0, 1, 2, 3, \ldots$). Having calculated the residues at the indicated points, we come to the power series

$$\int_0^\infty \Delta(\beta) \cos \beta y \, d\beta = \frac{\pi}{6a^2} \sum_{m=0}^\infty (-1)^m \left\{ \frac{4}{\sqrt{3}} \left[\frac{\eta^{6m}}{(6m)!} - \frac{\eta^{6m+4}}{(6m+4)!} \right] + \right.$$

$$+ 3\frac{|\eta|^{6m+5}}{(6m+5)\,!} + \frac{6}{\pi}\frac{\eta^{6m+2}}{(6m+2)\,!}\,[\ln|\eta| - \psi(6m+3)]\Bigg\}. \tag{1.6}$$

Here $\eta = ay$, and $\psi(z)$ is the psi-function, whose calculation is reduced to the recursion formula [7]

$$\psi(z+1) = \psi(z) + z^{-1}\,;\quad \psi(1) \approx -0.577216\,. \tag{1.7}$$

The remaining integrals are taken in a similar way, with the following expression being used for the sine:

$$\sin\beta y = -\frac{1}{4i}\int_{c-i\infty}^{c+i\infty}\frac{(\beta y)^z\,dz}{\Gamma(1+z)\cos\frac{\pi z}{2}}\,;\quad \frac{1}{2} < c < 1\,;\quad \mathrm{Re}\, z \geq c\,.$$

As a result, we have

$$\int_0^\infty \beta\Delta(\beta)\cos\beta y\,d\beta = \frac{\pi}{6a}\sum_{m=0}^{\infty}(-1)^m\left\{\frac{4}{\sqrt{3}}\left[\frac{\eta^{6m}}{(6m)\,!} + \frac{\eta^{6m+2}}{(6m+2)\,!}\right] - \right.$$

$$\left. - 3\frac{|\eta|^{6m+1}}{(6m+1)\,!} + \frac{6}{\pi}\frac{\eta^{6m+4}}{(6m+4)\,!}\,[\ln|\eta| - \psi(6m+5)]\right\}, \tag{1.8}$$

$$\int_0^\infty \Delta(\beta)\sin\beta y\,d\beta = \frac{\pi}{6a^2}\sum_{m=0}^{\infty}(-1)^m\left\{\frac{4}{\sqrt{3}}\left[\frac{\eta^{6m+1}}{(6m+1)\,!} + \frac{\eta^{6m+3}}{(6m+3)\,!}\right] - \right.$$

$$\left. - 3\frac{\eta^{6m+2}}{(6m+2)\,!}\,\mathrm{sgn}\,\eta + \frac{6}{\pi}\frac{\eta^{6m+5}}{(6m+5)\,!}\,[\ln|\eta| - \psi(6m+6)]\right\},$$

$$\int_0^\infty \beta^2\Delta(\beta)\sin\beta y\,d\beta = -\frac{\pi}{6}\sum_{m=0}^{\infty}(-1)^m\left\{\frac{4}{\sqrt{3}}\left[\frac{\eta^{6m+1}}{(6m+1)\,!} - \frac{\eta^{6m+5}}{(6m+5)\,!}\right] - \right.$$

$$\left. - 3\frac{\eta^{6m}}{(6m)\,!}\,\mathrm{sgn}\,y + \frac{6}{\pi}\frac{\eta^{6m+3}}{(6m+3)\,!}\,[\ln|\eta| - \psi(6m+4)]\right\}.$$

Now, we determine the region of convergence of the derived expansions. Applying the d'Alembert criterion to the terms of poorest convergence and taking into account the properties of factorials and formula (1.7), we find

$$\eta^6\lim_{m\to\infty}\frac{\psi(6m+k+6)\,(6m+k)\,!}{(6m+k+6)\,!\,\psi(6m+k)} = 0\,,$$

$$k \in \{3, 4, 5, 6\}\,,\quad \eta \in\,]-\infty\,;\,\infty[\,.$$

This means that the series obtained converge on the entire number axis. Replacing η by $a(y - y_1)$ in Eqs. (1.6) and (1.8) and substituting them into formulas (1.1), we obtain closed expressions for Green's functions without quadratures. Here y_1 is the coordinate of the load point, and y is a running variable.

2. Asymptotic Representations. For small values of the dimensionless parameter η it is possible to calculate just the first few terms in the series. If $|\eta| \leq 4$, sufficient accuracy is ensured already for $m \leq 3$. The approximate solution is reduced to the elementary formulas

$$q(y) = Paq_0(\eta)\,;\quad M(y) = Pa^{-1}M_0(\eta)\,;\quad Q(y) = PQ_0(\eta)\,, \tag{2.1}$$

TABLE 1. Asymptotic Values of q_0, M_0, and Q_0 Calculated by Formulas (2.2), (2.3) and Taken from [2]

η	$10q_0$	$10M_0$	$10Q_0$	$10q_0'$	$10M_0'$	$10Q_0'$
			Calculations by Eqs. (2.2)			
0	3.849	3.849	5.000	3.8	3.8	5.0
1	2.256	0.569	1.843	2.3	0.6	1.8
2	0.842	−0.416	0.364	0.8	−0.4	0.4
3	0.189	−0.494	0.013	0.2	−0.5	0.01
4	−0.023	−0.346	−0.161	0.0	−0.3	−0.2
			Calculations by Eqs. (2.3)			
0	3.85	3.85	5.000	3.8	3.8	5.0
0.2	3.69	2.93	4.24	3.7	2.9	4.2
0.4	3.38	2.15	3.53	3.4	2.1	3.5
0.6	3.01	1.51	2.90	3.0	1.5	2.9
0.8	2.63	0.99	2.33	2.6	1.0	2.3
1.0	2.26	0.57	1.85	2.3	0.6	1.8

$$q_0(\eta) = \frac{1}{6}\left\{\frac{4}{\sqrt{3}}\left[1 - \frac{\eta^4}{24} - \frac{\eta^6}{720} + \frac{\eta^{10}}{10!} + \frac{\eta^{12}}{12!} - \frac{\eta^{16}}{16!}\right] + \right.$$
$$+ 3\left(\frac{|\eta|^5}{120} - \frac{|\eta|^{11}}{11!}\right) + \frac{6}{\pi}\left[\frac{\eta^2}{2}(\ln|\eta| - 0.9228) - \right.$$
$$\left.\left. - \frac{\eta^8}{8!}(\ln|\eta| - 2.1406) + \frac{\eta^{14}}{14!}(\ln|\eta| - 2.6743)\right]\right\};$$

$$M_0(\eta) = \frac{1}{6}\left\{\frac{4}{\sqrt{3}}\left[1 + \frac{\eta^2}{2} - \frac{\eta^6}{720} - \frac{\eta^8}{8!} + \frac{\eta^{12}}{12!} + \frac{\eta^{14}}{14!}\right] - \right.$$
$$- 3\left(|\eta| - \frac{|\eta|^7}{7!} + \frac{|\eta|^{13}}{13!}\right) + \frac{6}{\pi}\left[\frac{\eta^4}{24}(\ln|\eta| - 1.5061) - \right.$$
$$\left.\left. - \frac{\eta^{10}}{10!}(\ln|\eta| - 2.3517) + \frac{\eta^{16}}{16!}(\ln|\eta| - 2.8035)\right]\right\}; \tag{2.2}$$

$$Q_0(\eta) = -\frac{1}{6}\left\{\frac{4}{\sqrt{3}}\left[\eta - \frac{\eta^5}{120} - \frac{\eta^7}{7!} + \frac{\eta^{11}}{11!} + \frac{\eta^{13}}{13!} - \frac{\eta^{17}}{17!}\right] - \right.$$
$$- 3\,\mathrm{sgn}\,\eta\left(1 - \frac{\eta^6}{720} + \frac{\eta^{12}}{12!}\right) + \frac{6}{\pi}\left[\frac{\eta^3}{6}(\ln|\eta| - 1.2561) - \right.$$
$$\left.\left. - \frac{\eta^9}{9!}(\ln|\eta| - 2.2518) + \frac{\eta^{15}}{15!}(\ln|\eta| - 2.7410)\right]\right\}.$$

Since the series are alternating, then, according to the Leibnitz theorem, the absolute values of their residuals do not exceed those of the first discarded terms. Therefore, the error of formulas (2.2) is smaller in absolute value than the sum of the absolute values of the terms in Eqs. (1.6) and (1.8) that correspond to $m = 4$.

In order to verify that formulas (2.2) are highly accurate at small values of η, we consider the numerical results in Table 1, where asymptotic values of q_0, M_0, and Q_0 are given. The primed values of the same quantities are taken from Table 10 of [2]. A comparison shows good agreement of them over the entire considered range of the parameter η. When $\eta \leq 1$, it is sufficient to retain in formulas (2.2) terms whose order of magnitude is lower than $O(\eta^6)$, which leads to the form

$$q_0(\eta) = \frac{1}{6}\left[\frac{4}{\sqrt{3}}\left(1 - \frac{\eta^4}{24}\right) + \frac{|\eta|^5}{40} + \frac{3\eta^2}{\pi}(\ln|\eta| - 0.9228)\right],$$

$$M_0(\eta) = \frac{1}{6}\left[\frac{4}{\sqrt{3}}\left(1 + \frac{\eta^2}{2}\right) - 3|\eta| + \frac{\eta^4}{4\pi}(\ln|\eta| - 1.5061)\right], \tag{2.3}$$

$$Q_0(\eta) = -\frac{1}{6}\left[\frac{4}{\sqrt{3}}\eta - 3\,\mathrm{sgn}\,\eta + \frac{\eta^3}{\pi}(\ln|\eta| - 1.2561)\right].$$

Results of calculations carried out by formulas (2.3) for dimensionless contact forces, bending moments, and intersecting forces are also given in Table 1. The primed values [2] are obtained by means of numerical integration of quadratures. A comparison shows that for $\eta \leq 1$ formulas (2.3) provide good accuracy of calculation.

At the point of application of the force expressions (2.1) and (2.3) yield

$$q(0) = \frac{2P}{3\sqrt{3}}\left(\frac{Eh}{2E_1I_1}\right)^{1/3}; \quad M(0) = \frac{2P}{3\sqrt{3}}\left(\frac{Eh}{2E_1I_1}\right)^{-1/3},$$

which agrees with the solutions of [1]. The intersecting force $Q = (P/2)\,\mathrm{sgn}\,\eta$ suffers a discontinuity of magnitude P, which corresponds to physical concepts.

Thus, in the zone of application of a force the stressed state of a rib can be calculated by using simple closed formulas.

NOTATION

P, concentrated force transmitted to the plane body through the rib; E_1I_1, bending stiffness of the rib in the plane of action of the force; x, coordinate in a direction perpendicular to the rib; y, coordinate in a direction parallel to the edge of the plane body; $q(y)$, contact force of interaction of the rib and the plane body; Q, M, intersecting force and bending moment in the rib; h, thickness of the plane body; E, ν, elasticity modulus and Poisson coefficient of the material of the plane body; p_0, pressure at infinity that balances the concentrated force P.

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